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Inversion method approach to the quantum kinetic equation: inhomogeneous case

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Abstract

The inversion method to derive the quantum kinetic equation is extended to the case of an inhomogeneous system. Within the closed-time-path formalism, the source is introduced to probe the (Fourier transform of the) Wigner distribution function and the equation of motion for the Wigner function is derived perturbatively by the inversion formulae. The equation of motion has the form of the generalized Boltzmann equation, which is non-local in space and time. For the obtained kinetic equation, the local conservation laws of number, energy and momentum are confirmed.

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1. Introduction

Since the theory of Kadanoff and Baym [1], many works have been performed based on the Green function approach to derive the quantum kinetic equation (QKE) or the generalized Boltzmann equation. Most of these works rely on the Kadanoff–Baym ansatz [1] or its generalized version (GKB ansatz) [2]. The basis of the ansatz, however, is not completely clarified and hence the definitions of the particle in these theories are not clear. An alternative approach is the counter-term method: in the framework of the closed-time-path (CTP) formalism [3, 4] or of the thermo-field dynamics [5], the counter-term is subtracted from the 2×2-Lagrangian and the self-consistent condition is adopted to determine the counter-term. The self-consistent condition leads to the QKE. This approach is free from the ansatz, but the ambiguity remains in the choice of the self-consistent condition.

In the preceding papers [6, 7], we presented a new approach to derive the QKE using the inversion method. In this approach, we first calculate the expectation value of the number operator under the existence of the probing source, and obtain the number as a functional of the source. The obtained functional relation is inverted to express the source as a functional of the number, and then the equation of motion (EoM) of the number is obtained by removing the source. The method of introducing the source is a crucial point in the method and the source we have used is motivated by the dissipative counter-term of Lawrie [3]. Such an approach has an advantage that, since the expectation value of the number operator is directly calculable at

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the first step, the number appearing in the QKE is well defined, and that the EoM is determined by simply removing the source after the inversion, without arbitrariness of the condition.

In this paper, we extend the method to the case of a system prepared with the inhomogeneous density matrix at some initial time. A systematic investigation of the QKE in the inhomogeneous situation can be seen in [8], where the extension of the thermo-field dynamics in the homogeneous case [5] to the inhomogeneous one is discussed and the Markovian QKE is obtained. Our previous results [6, 7] can similarly be extended to the inhomogeneous case, but the resulting QKEs become non-Markovian. Moreover in our method, systematic evaluation of the conserved currents can be performed.

After a brief description of inversion method in section 2, it is applied in section 3 to the system which is initially inhomogeneous. We introduce a local probing source for the (Fourier transform of) the Wigner distribution function (WDF) by extending the source of [6] to the two-point form. Then we will find that the inversion method is straightforwardly applicable. The calculations are carried out up to the second order in interaction and the obtained EoM of the WDF has the form of a QKE which is non-local in both time and space. The QKE is reduced to a more familiar form in the weakly inhomogeneous case where the gradient expansion can be carried out.

In section 4, the local conservation laws of number, energy and momentum densities are confirmed. From the symmetries of the Lagrangian, conserved current is derived by Noether's theorem, and it is expressed as functional of the WDF with the aid of inversion in section 3. The QKE obtained in section 3 turns out to guarantee explicitly the conservation of these currents.

2. Inversion method

The inversion method [9, 10] is a simple technique to obtain the EoM of expectation values of an arbitrary operator. For any operator $\hat{Q}$ of interest, the expectation value $Q(t)$ at the time t is first calculated under the existence of some probing source J, which is time dependent in general. Then $Q(t)$ becomes a functional of J and a function of t; $Q(t) = f\,[t; J]$. By solving the relation inversely for J, the source J is expressed as a functional of Q and a function of t; $J(t) = g\,[t; Q]$. The removal of the source at this stage yields an equation to determine the temporal change of Q, i.e. the EoM. This inversion process can be carried out perturbatively using the following 'inversion formulae' [10].

Assume the perturbative expansion of Q as a functional of J; its inverted series are also obtained perturbatively,

$$Q(t) = f[t; J] = \sum_n \lambda^n f_n[t; J] \qquad J(t) = g[t; Q] = \sum_m \lambda^m g_m[t; Q] \tag{1}$$

where λ is a small parameter of the expansion. Then in an identity $Q(t) = f\,[t; g\,[Q]]$, the rhs is expanded in terms of λ, and comparing both sides in each order of λ the relations among g_ms and f_ns are obtained, which leads to the 'inversion formulae':

$$g_0[t; Q] = f_0^{-1}[t; Q] \tag{2}$$

$$g_1[t; Q] = -\int \mathrm{d}t'\, f_0^{(1)^{-1}}[t, t'; Q]\, f_1[t'; g_0] \tag{3}$$

$$g_2[t; Q] = -\int \mathrm{d}t'\, f_0^{(1)^{-1}}[t, t'; Q] \left(\tfrac{1}{2} \int \mathrm{d}s\, \mathrm{d}s'\, f_0^{(2)}[t', s, s'; Q]\, g_1[s; Q]\, g_1[s'; Q] + \int \mathrm{d}s\, f_1^{(1)}[t', s; Q]\, g_1[s; Q] + f_2[t'; g_0[Q]] \right) \tag{4}$$

where we have used the abbreviations for $k = 1, 2, \ldots,$

$$f_n^{(k)}[t, s_1, s_2, \ldots, s_k; Q] \equiv \left. \frac{\delta^k f_n[t; J]}{\delta J(s_1)\delta J(s_2)\cdots\delta J(s_k)} \right|_{J=g_0[Q]} \tag{5}$$

and $f_0^{(1)-1}$ is defined by $\int \mathrm{d}t'\, f_0^{(1)-1}[t, t'; Q] f_0^{(1)}[t', s; Q] = \delta(t - s)$.

In the following, we apply the inversion method to derive the EoM of the WDF. There, the inversion formulae are slightly extended to include wavenumber indices in addition to the time arguments.

3. Derivation of QKEs: generalized Boltzmann equation

3.1. Introduction of probing source

The system to be considered is the same as in [6, 7]: a non-relativistic boson field described by the Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}_{\mathrm{int}}$ with

$$\hat{H}_0 = \sum_{\boldsymbol{k}} \epsilon_{\boldsymbol{k}} \hat{\psi}_{\boldsymbol{k}}^\dagger \hat{\psi}_{\boldsymbol{k}} \qquad \hat{H}_{\mathrm{int}} = \frac{\lambda}{4} \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{q}} \hat{\psi}_{\boldsymbol{k}+\boldsymbol{q}}^\dagger \hat{\psi}_{\boldsymbol{k}'-\boldsymbol{q}}^\dagger \hat{\psi}_{\boldsymbol{k}} \hat{\psi}_{\boldsymbol{k}'}. \tag{6}$$

What we are going to derive is the EoM of the WDF

$$f_{\boldsymbol{K}}(\boldsymbol{X}, t) = \int \frac{\mathrm{d}\Delta\boldsymbol{x}}{V} \mathrm{e}^{-\mathrm{i}\boldsymbol{K}\cdot\Delta\boldsymbol{x}} \left\langle \hat{\psi}^\dagger \left(\boldsymbol{X} - \frac{\Delta\boldsymbol{x}}{2}, t \right) \hat{\psi} \left(\boldsymbol{X} + \frac{\Delta\boldsymbol{x}}{2}, t \right) \right\rangle \tag{7}$$

where $\hat{\psi}(\boldsymbol{x}) = \frac{1}{\sqrt{V}} \sum_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \hat{\psi}_{\boldsymbol{k}}$ and the angular bracket implies the average over the initial density matrix; $\langle \cdots \rangle = \mathrm{Tr}\, \hat{\rho} \cdots$. The problem is equivalent to deriving the EoM of the correlation function defined as the Fourier transform of the WDF:

$$z_{\boldsymbol{k},\boldsymbol{q}}(t) \equiv \left\langle \hat{\psi}_{\boldsymbol{q}}^\dagger(t) \hat{\psi}_{\boldsymbol{k}}(t) \right\rangle = \int \mathrm{d}\boldsymbol{x}\, \mathrm{e}^{-\mathrm{i}(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{x}} f_{\frac{\boldsymbol{k}+\boldsymbol{q}}{2}}(\boldsymbol{x}, t). \tag{8}$$

By the second equality, the EoM of z can easily be reduced to that of the WDF (7), and we will see that the EoM becomes rather simple when we work with z.

Equation (8) can be represented by path integration as

$$z_{\boldsymbol{k},\boldsymbol{q}}(t) \propto \int [\mathrm{d}\psi_1\, \mathrm{d}\psi_2]\, \psi_{\boldsymbol{q}}^*(t) \psi_{\boldsymbol{k}}(t) \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int_{\mathrm{I}}^{\mathrm{f}} \mathrm{d}s\, (L(\psi_1) - L(\psi_2))} \left\langle \psi_{1,\mathrm{I}} \left| \hat{\rho} \right| \psi_{2,\mathrm{I}} \right\rangle \tag{9}$$

where L is the Lagrangian corresponding to the Hamiltonian H and ψ_1 and ψ_2 are respectively introduced as integration variables due to the forward and backward time evolution operators, and satisfy the boundary conditions $\psi_2(t_{\mathrm{I}}) = \psi_{2,\mathrm{I}}$ and $\psi_1(t_{\mathrm{I}}) = \psi_{1,\mathrm{I}}$. This type of formulation is called the CTP formalism [11, 12].

In this paper, the initial distribution $\hat{\rho}$ is assumed to be inhomogeneous, but, for simplicity, no correlation among the fields is considered. The initial correlations can be treated similarly as in [7] by introducing the imaginary time path.

In order to consider an inhomogeneous system, we introduce the local probing source throughout the time evolution in (9). As in the case of a homogeneous system [6, 7], if we introduce it naively in the CTP formalism [10, 12], we are faced with the difficulty that z does not depend on J in the lowest order of perturbation. To overcome this problem, according to the method in [6], we first write the free part of the Lagrangian in (9) in the quadratic form $L_0(\psi_1) - L_0(\psi_2) = \sum_{ij} \psi_{k,i}^* \mathcal{D}_{\boldsymbol{k}\boldsymbol{q},ij} \psi_{q,j}$, and then the source $J_{k,q}$ is built into the matrix $\mathcal{D}_{\boldsymbol{k},\boldsymbol{q}}$ as

$$\mathcal{D}_{\boldsymbol{k},\boldsymbol{q}} = \begin{pmatrix} (\mathrm{i}\hbar\partial_t - \epsilon_{\boldsymbol{k}})\delta_{\boldsymbol{k},\boldsymbol{q}} + \mathrm{i}J_{\boldsymbol{k},\boldsymbol{q}}(t) & -\mathrm{i}J_{\boldsymbol{k},\boldsymbol{q}}(t) \\ -\mathrm{i}J_{\boldsymbol{k},\boldsymbol{q}}(t) & -(\mathrm{i}\hbar\partial_t - \epsilon_{\boldsymbol{k}})\delta_{\boldsymbol{k},\boldsymbol{q}} + \mathrm{i}J_{\boldsymbol{k},\boldsymbol{q}}(t) \end{pmatrix}. \tag{10}$$

Now $z_{k,q}$ and $J_{k,q}$ play the role of Q and J of (1), respectively. Thus we calculate the unperturbed propagator, which is essentially the inverse of (10), to construct the perturbative expansion of (9).

3.2. Unperturbed propagator

In general, the Green function in CTP formalism can be represented by the following 2×2 matrix form:

$$G_{k,q}(t,s) \equiv -\operatorname{Tr}\,\hat{\rho}\begin{pmatrix} T\hat{\psi}_k(t)\hat{\psi}_q^\dagger(s) & \hat{\psi}_q^\dagger(s)\hat{\psi}_k(t) \\ \hat{\psi}_k(t)\hat{\psi}_q^\dagger(s) & \tilde{T}\hat{\psi}_k(t)\hat{\psi}_q^\dagger(s)\end{pmatrix}_{\mathrm{c}}$$
$$= \theta(t-s)\begin{pmatrix} h_{k,q}(t,s) & k_{k,q}(t,s) \\ h_{k,q}(t,s) & k_{k,q}(t,s)\end{pmatrix} + \theta(s-t)\begin{pmatrix} k^*_{q,k}(s,t) & k^*_{q,k}(s,t) \\ h^*_{q,k}(s,t) & h^*_{q,k}(s,t)\end{pmatrix} \tag{11}$$

where the subscript 'c' implies the connected part and

$$h_{k,q}(t,s) \equiv -\langle\hat{\psi}_k(t)\hat{\psi}_q^\dagger(s)\rangle_{\mathrm{c}} \qquad k_{k,q}(t,s) \equiv -\langle\hat{\psi}_q^\dagger(s)\hat{\psi}_k(t)\rangle_{\mathrm{c}}. \tag{12}$$

As in [6], we search for the unperturbed propagator $G^{(0)}$ with the form (11) under the existence of the probing source: the source of (10) is constructed as to keep this structure [7].

Starting from (10), the unperturbed propagator is calculated from the relation

$$\sum_{k'} \mathcal{D}_{k,k'}(t,\partial_t) G^{(0)}_{k',q}(t,s) = \sum_{k'} G^{(0)}_{q,k'}(t,s)\mathcal{D}_{k',q}(s,-\overleftarrow{\partial}_s) \tag{13}$$
$$= -\mathrm{i}\hbar\delta_{k,q}\delta(t-s) \tag{14}$$

where $\overleftarrow{\partial}$ is the differentiation acting to the left. Assuming that $G^{(0)}$ has the same structure as (11) in which h and k are replaced by $h^{(0)}$ and $k^{(0)}$, respectively, equation (13) for $t > s$ leads to

$$(\mathrm{i}\hbar\partial_t - \epsilon_k)\, h^{(0)}_{k,q}(t,s) = 0 \qquad (\mathrm{i}\hbar\partial_t - \epsilon_k)\, k^{(0)}_{k,q}(t,s) = 0 \tag{15}$$

and for $s > t$ to

$$(\mathrm{i}\hbar\partial_t - \epsilon_k)\, h^{(0)*}_{q,k}(s,t) = \mathrm{i}\sum_{k'} J_{k,k'}(t)\left\{h^{(0)*}_{q,k'}(s,t) - k^{(0)*}_{q,k'}(s,t)\right\} \tag{16}$$

$$(\mathrm{i}\hbar\partial_t - \epsilon_k)\, k^{(0)*}_{q,k}(s,t) = -\mathrm{i}\sum_{k'} J_{k,k'}(t)\left\{h^{(0)*}_{q,k'}(s,t) - k^{(0)*}_{q,k'}(s,t)\right\} \tag{17}$$

with the boundary conditions at $t = s$ given as

$$h^{(0)}_{k,q}(s,s) - h^{(0)*}_{q,k}(s,s) = 0 \qquad h^{(0)}_{k,q}(s,s) - k^{(0)*}_{q,k}(s,s) = -\delta_{k,q} \tag{18}$$

$$k^{(0)}_{k,q}(s,s) - k^{(0)*}_{q,k}(s,s) = 0 \qquad k^{(0)}_{k,q}(s,s) - h^{(0)*}_{q,k}(s,s) = \delta_{k,q}. \tag{19}$$

From the definition of z in (8) and from the conditions (18) and (19), we can write

$$h^{(0)}_{k,q}(s,s) = -\bar{z}^{(0)}_{k,q}(s) \qquad k^{(0)}_{k,q}(s,s) = -z^{(0)}_{k,q}(s) \tag{20}$$

where $z^{(0)}_{k,q}$ is the unperturbed part of the correlation function (8) and $\bar{z}^{(0)}_{k,q} = z^{(0)}_{k,q} + \delta_{k,q}$. Note that $z^*_{q,k} = z_{k,q}$ holds since $\hat{\rho}$ is Hermitian.

Equations in (15) can be solved with the results

$$h^{(0)}_{k,q}(t,s) = -\mathrm{e}^{-\mathrm{i}\omega_k(t-s)}\bar{z}^{(0)}_{k,q}(s) \tag{21}$$

$$k^{(0)}_{k,q}(t,s) = -\mathrm{e}^{-\mathrm{i}\omega_k(t-s)}z^{(0)}_{k,q}(s) \tag{22}$$

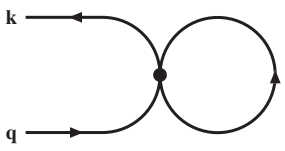

Figure 1. Diagram for the first-order contributions to $z_{k,q}[J]$.

for $t > s$, where $\omega_k \equiv \epsilon_k/\hbar$. Then $h^{(0)*}_{q,k}(s,t)$ and $k^{(0)*}_{q,k}(s,t)$ are obtained for $s > t$ by exchanging $t, \boldsymbol{k}$ and $s, \boldsymbol{q}$ in the complex conjugations of (21) and (22). Substituting them into (16) or (17), the two equations turn out to be identical, and lead to the EoM of $z^{(0)}_{k,q}$:

$$J_{k,q}(t) = \left\{\hbar\partial_t + \mathrm{i}(\epsilon_k - \epsilon_q)\right\} z^{(0)}_{k,q}(t). \tag{23}$$

This can be solved as

$$z^{(0)}_{k,q}[t;J] = \mathrm{e}^{-\mathrm{i}(\omega_k-\omega_q)(t-s)} z^{(0)}_{k,q}(t_1) + \frac{1}{\hbar}\int_{t_1}^{t} \mathrm{d}s\, \mathrm{e}^{-\mathrm{i}(\omega_k-\omega_q)(t-s)} J_{k,q}(s) \tag{24}$$

and gives the expression of z as a functional of J in the zeroth order of the interaction. Thus equations (21), (22) and (24) determine the unperturbed propagator $G^{(0)}$ if we substitute $h^{(0)}$ and $k^{(0)}$ into (11) in place of h and k, respectively. Equation (24) corresponds to $f_0[t;J]$ of (1), and (23) to the inversion (2). Note that, although (22) and (21) have similar form to the GKB ansatz, they are derived without any *ad hoc* assumption.

3.3. First-order correction: mean-field effect

With the propagator $-G^{(0)}$ expressed by the arrow going from ψ^* to ψ and the vertex $\pm\lambda/\mathrm{i}\hbar$ (the signs + and − respectively correspond to the forward and backward time-path), the first-order correction $z^{(1)}$ is represented by the tadpole-type diagram shown in figure 1 and calculated as

$$z^{(1)}_{k,q}(t) = \frac{\lambda}{\mathrm{i}\hbar}\sum_{q',k',m}\int_{t_1}^{t}\mathrm{d}s\, \mathrm{e}^{-\mathrm{i}(\omega_k-\omega_q)(t-s)}\left\{\bar{z}^{(0)}_{k,k'} z^{(0)*}_{q,q'} z^{(0)*}_{m-k',m-q'} - z^{(0)}_{k,k'}\bar{z}^{(0)*}_{q,q'} z^{(0)*}_{m-k',m-q'}\right\}(s) \tag{25}$$

$$= \frac{\lambda}{\mathrm{i}\hbar}\sum_{q',m}\int_{t_1}^{t}\mathrm{d}s\, \mathrm{e}^{-\mathrm{i}(\omega_k-\omega_q)(t-s)}\left\{z^{(0)*}_{q,q'} z^{(0)*}_{m-k,m-q'} - z^{(0)}_{k,q'} z^{(0)*}_{m-q',m-q}\right\}(s). \tag{26}$$

In the second equality, the summation index $\boldsymbol{k}'$ is changed into $\boldsymbol{q}'$ for the second term of the integrand. Note that this is a functional of J since the $z^{(0)}$s in the rhs are the functionals given in (24).

With the inversion formula (3), equation (23) receives the correction up to the first order of interaction as

$$J_{k,q}(t) = \left\{\hbar\partial_t + \mathrm{i}(\epsilon_k - \epsilon_q)\right\} z_{k,q}(t) + \mathrm{i}\lambda\sum_{q',m}\left\{z^*_{q,q'} z^*_{m-k,m-q'} - z_{k,q'} z_{m-q,m-q'}\right\}(t). \tag{27}$$

This can be written in terms of the WDF with the use of (8), and by removing the source the EoM of the WDF with the correction of $\mathrm{O}(\lambda)$ is obtained as

$$\left\{\hbar\partial_t + \mathrm{i}(\epsilon_k - \epsilon_q)\right\}\int \mathrm{d}\boldsymbol{x}\, \mathrm{e}^{-\mathrm{i}\Delta\boldsymbol{k}\cdot\boldsymbol{x}} f_K(\boldsymbol{x},t)$$

$$= -\mathrm{i}\lambda\sum_{q'}\int \mathrm{d}\boldsymbol{x}_1\mathrm{d}\boldsymbol{x}_2\Big\{\mathrm{e}^{\mathrm{i}(q-q')\cdot x_1+(q'-k)\cdot x_2} f_{\frac{q+q'}{2}}(\boldsymbol{x}_1,t) n(\boldsymbol{x}_2,t)$$

$$-\mathrm{e}^{-\mathrm{i}(k-q')\cdot x_1+(q'-q)\cdot x_2} f_{\frac{k+q'}{2}}(\boldsymbol{x}_1,t) n(\boldsymbol{x}_2,t)\Big\} \tag{28}$$

where $\boldsymbol{K} = (\boldsymbol{k}+\boldsymbol{q})/2$, $\Delta\boldsymbol{k} = \boldsymbol{k}-\boldsymbol{q}$ and $n(\boldsymbol{x}) = \sum_{\boldsymbol{k}} f_{\boldsymbol{k}}(\boldsymbol{x})$ is the local particle density, which is produced after the summation over $\boldsymbol{m}$ in (27).

The kinetic equation (28) is reduced to a more familiar form by the gradient expansion. It is expected that the summation over $\boldsymbol{q}'$ in (28) will provide the spatial decay with some characteristic length. Then if the spatial variation of the WDF has much longer scale than the characteristic length, we can expand the WDF in (28) around $\boldsymbol{X} = (\boldsymbol{x}_1+\boldsymbol{x}_2)/2$ in powers of $\Delta\boldsymbol{x} = \boldsymbol{x}_1 - \boldsymbol{x}_2$, and take the terms up to the first order in the gradient. Then the rhs of (28) becomes

$$
\begin{aligned}
-\mathrm{i}\lambda \sum_{\boldsymbol{q}'} \int \mathrm{d}\boldsymbol{X}\, \mathrm{d}\Delta\boldsymbol{x} \Bigg\{ & \mathrm{e}^{-\mathrm{i}\{\Delta\boldsymbol{k}\cdot\boldsymbol{X}-(\boldsymbol{K}-\boldsymbol{q}')\cdot\Delta\boldsymbol{x}\}} \left(f_{\frac{\boldsymbol{q}+\boldsymbol{q}'}{2}}(\boldsymbol{X},t) \left(1 - \frac{\Delta\boldsymbol{x}}{2}\cdot \overleftrightarrow{\partial}_{\boldsymbol{X}}\right) n(\boldsymbol{X},t)\right) \\
& - \mathrm{e}^{-\mathrm{i}\{\Delta\boldsymbol{k}\cdot\boldsymbol{X}+(\boldsymbol{K}-\boldsymbol{q}')\cdot\Delta\boldsymbol{x}\}} \left(f_{\frac{\boldsymbol{k}+\boldsymbol{q}'}{2}}(\boldsymbol{X},t) \left(1 - \frac{\Delta\boldsymbol{x}}{2}\cdot \overleftrightarrow{\partial}_{\boldsymbol{X}}\right) n(\boldsymbol{X},t)\right) \Bigg\} \\
= & -\mathrm{i}\lambda V \int \mathrm{d}\boldsymbol{X}\, \mathrm{e}^{-\mathrm{i}\Delta\boldsymbol{k}\cdot\boldsymbol{X}} \Bigg\{ \left(f_{\boldsymbol{K}-\frac{\Delta\boldsymbol{k}}{4}} - f_{\boldsymbol{K}+\frac{\Delta\boldsymbol{k}}{4}}\right) n \\
& + \frac{\mathrm{i}}{4}\partial_{\boldsymbol{K}} \cdot \left(\left(f_{\boldsymbol{K}-\frac{\Delta\boldsymbol{k}}{4}} + f_{\boldsymbol{K}+\frac{\Delta\boldsymbol{k}}{4}}\right) \overleftrightarrow{\partial}_{\boldsymbol{X}}\, n\right) \Bigg\}
\end{aligned}
\tag{29}
$$

where $\overleftrightarrow{\partial}_{\boldsymbol{X}}$ is defined by $\xi\, \overleftrightarrow{\partial}_{\boldsymbol{X}}\, \eta = \xi(\partial_{\boldsymbol{X}}\eta) - (\partial_{\boldsymbol{X}}\xi)\eta$, and in the equality of (29), the integration over $\Delta\boldsymbol{x}$ is first carried out and then the summation over $\boldsymbol{q}'$.

Then we carry out the expansion around $\Delta\boldsymbol{k} = \boldsymbol{0}$ and take up to the first order of $\partial_{\boldsymbol{X}}$ after the partial integration. This produces terms proportional to $(\partial_{\boldsymbol{K}}\cdot\partial_{\boldsymbol{X}} f)n$ and to $(\partial_{\boldsymbol{K}} f)\cdot(\partial_{\boldsymbol{X}} n)$, but the former contributions from the first and second terms in the curly brackets of (29) cancel each other and only the latter terms remain. As a result, equation (28) is reduced to

$$
\left\{\partial_t + \boldsymbol{v}^{\mathrm{g}}_{\boldsymbol{K}}\cdot\partial_{\boldsymbol{X}}\right\} f_{\boldsymbol{K}}(\boldsymbol{X},t) = \frac{\lambda V}{\hbar}\left(\partial_{\boldsymbol{X}} n(\boldsymbol{X},t)\right)\cdot\partial_{\boldsymbol{K}} f_{\boldsymbol{K}}(\boldsymbol{X},t) \tag{30}
$$

where the group velocity is defined by $\boldsymbol{v}^{\mathrm{g}}_{\boldsymbol{K}} = \partial_{\boldsymbol{K}}\epsilon_{\boldsymbol{K}}/\hbar$. The rhs of (30) is the first-order correction of the gradient expansion, and apparently it expresses the effect of the mean-field potential force. In the literature (e.g. [1]), the mean-field potential is usually included in the effective one-particle energy, but in our case, since the one-particle energy ϵ is position independent, the mean-field potential term appears as in (30).

3.4. Second-order correction: collision term

The second-order corrections to $z_{\boldsymbol{k},\boldsymbol{q}}$ come from the diagram shown in figure 2(*a*) and from those with tadpoles given in figures 2(*b*) and (*c*). The diagrams with tadpoles, however, vanish after the inversion; in the inversion formula (4), the tadpole contributions to f_2 compensate for the $f_1^{(1)} g_1$. Thus the diagram in figure 2(*a*) is sufficient for the purpose of deriving the EoM, and is evaluated as

$$
\begin{aligned}
z^{(2)}_{\boldsymbol{k},\boldsymbol{q}}[t;J] = & \frac{1}{2}\left(\frac{\lambda}{\hbar}\right)^2 \sum_{\boldsymbol{l},\boldsymbol{m}} \int_{t_1}^{t} \mathrm{d}s\; \mathrm{e}^{-\mathrm{i}(\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{q}})(t-s)} \\
& \times \int_{t_1}^{t} \mathrm{d}s' \left\{ \mathrm{e}^{\mathrm{i}\omega^{(2)}_{\boldsymbol{q},\boldsymbol{k},\boldsymbol{l},\boldsymbol{m}}(s-s')} \tilde{Z}^{(2)}_{\boldsymbol{q},\boldsymbol{k},\boldsymbol{l},\boldsymbol{m}}(s') + \mathrm{e}^{-\mathrm{i}\omega^{(2)}_{\boldsymbol{k},\boldsymbol{q},\boldsymbol{l},\boldsymbol{m}}(s-s')} \tilde{Z}^{(2)*}_{\boldsymbol{k},\boldsymbol{q},\boldsymbol{l},\boldsymbol{m}}(s') \right\}
\end{aligned}
\tag{31}
$$

where $\omega^{(2)}_{\boldsymbol{q},\boldsymbol{k},\boldsymbol{l},\boldsymbol{m}} \equiv \omega_{\boldsymbol{q}} + \omega_{\boldsymbol{m}-\boldsymbol{k}} - \omega_{\boldsymbol{l}} - \omega_{\boldsymbol{m}-\boldsymbol{l}}$ and $\tilde{Z}^{(2)}$ is defined as

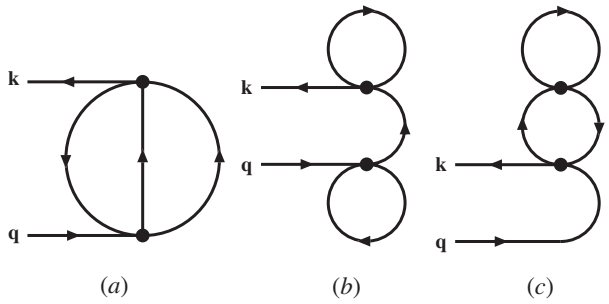

Figure 2. Diagrams for $O(\lambda^2)$ contributions to $z_{k,q}[J]$.

$$\tilde{Z}^{(2)}_{q,k,l,m} \equiv \sum_{q',l',m'} \left\{ \bar{z}^{(0)*}_{q,q'} \bar{z}^{(0)*}_{m-k,m'-q'} z^{(0)}_{l,l'} z^{(0)}_{m-l,m'-l'} - z^{(0)*}_{q,q'} z^{(0)*}_{m-k,m'-q'} \bar{z}^{(0)}_{l,l'} \bar{z}^{(0)}_{m-l,m'-l'} \right\}. \tag{32}$$

By the second-order inversion formula (4), the $O(\lambda^2)$ correction to (27) is derived. Since (24) is linear in J, $f_0^{(2)}$ in (4) vanishes, and, as mentioned above, the contributions from $f_1^{(1)} g_1$ cancel with diagrams (*b*) and (*c*) in figure 2. Thus the EoM of z becomes

$$\begin{aligned} \partial_t z_{k,q}(t) = {} & \frac{\epsilon_k - \epsilon_q}{\mathrm{i}\hbar} z_{k,q}(t) + \frac{\lambda}{\mathrm{i}\hbar} \sum_{q',m} \left\{ z^*_{q,q'} z^*_{m-k,m-q'} - z_{k,q'} z_{m-q,m-q'} \right\}(t) \\ & + \frac{1}{2} \left(\frac{\lambda}{\hbar} \right)^2 \sum_{l,m} \int_{t_1}^{t} \mathrm{d}s \left\{ \mathrm{e}^{\mathrm{i}\omega^{(2)}_{q,k,l,m}(t-s)} Z^{(2)}_{q,k,l,m}(s) + \mathrm{e}^{-\mathrm{i}\omega^{(2)}_{k,q,l,m}(t-s)} Z^{(2)*}_{k,q,l,m}(s) \right\} \end{aligned} \tag{33}$$

where $Z^{(2)}$ is obtained from $\tilde{Z}^{(2)}$ by replacing $z^{(0)}[J]$ in (32) with z. A similar (but Markovian) equation is obtained in [8] and ours is the non-Markovian extension of it.

Substituting (8) into (33), we obtain the EoM of the WDF with the second-order correction. Although the Born approximation in the case of the local interaction (6) is considered here, the obtained equation reveals the collision term, which is non-local in the WDF. This is just due to the fact that the non-local information is required in (7) to determine the WDF. A similar type of non-locality appears in [13, 14], where the density operator formalism is used, and in [8].

As in section 3.3, we can carry out the gradient expansion in the weakly inhomogeneous case. For example in Z, consider the term corresponding to the first term of (32). By rearranging the summation for $\Delta \boldsymbol{m} = \boldsymbol{m} - \boldsymbol{m}'$ and $\Delta \boldsymbol{l} = \boldsymbol{l} - \boldsymbol{l}'$, it is expressed as

$$\begin{aligned} & \int \mathrm{d}\boldsymbol{x}_1 \mathrm{d}\boldsymbol{x}_2 \, \mathrm{d}\boldsymbol{y}_1 \, \mathrm{d}\boldsymbol{y}_2 \sum_{\boldsymbol{q}', \Delta\boldsymbol{l}, \Delta\boldsymbol{m}} \mathrm{e}^{\mathrm{i}(\Delta\boldsymbol{m} - \Delta\boldsymbol{k})\cdot\boldsymbol{X} - \mathrm{i}\left(\frac{\Delta\boldsymbol{m}}{2} - \boldsymbol{K} + \boldsymbol{q}'\right)\cdot\Delta\boldsymbol{x}} \bar{f}_{\frac{\boldsymbol{q}+\boldsymbol{q}'}{2}}(\boldsymbol{x}_1) \bar{f}_{\boldsymbol{m} - \frac{\Delta\boldsymbol{m}+\boldsymbol{k}+\boldsymbol{q}'}{2}}(\boldsymbol{x}_2) \\ & \quad \times \mathrm{e}^{-\mathrm{i}\Delta\boldsymbol{m}\cdot\boldsymbol{Y} + \mathrm{i}\left(\frac{\Delta\boldsymbol{m}}{2} - \Delta\boldsymbol{l}\right)\cdot\Delta\boldsymbol{y}} f_{\boldsymbol{l} - \frac{\Delta\boldsymbol{l}}{2}}(\boldsymbol{y}_1) f_{\boldsymbol{m}-\boldsymbol{l} - \frac{\Delta\boldsymbol{m}-\Delta\boldsymbol{l}}{2}}(\boldsymbol{y}_2) \\ & \quad \simeq V^2 \int \mathrm{d}\boldsymbol{X} \mathrm{d}\boldsymbol{Y} \sum_{\Delta\boldsymbol{m}} \mathrm{e}^{\mathrm{i}(\Delta\boldsymbol{m} - \Delta\boldsymbol{k})\cdot\boldsymbol{X}} \\ & \quad \times \left\{ \bar{f}_{\boldsymbol{K} + \frac{\Delta\boldsymbol{k} - \Delta\boldsymbol{m}}{4}} \left(1 - \frac{\mathrm{i}}{4} \overleftrightarrow{\partial}_{\boldsymbol{x}} \cdot \overleftrightarrow{\partial}_{\boldsymbol{k}} \right) \bar{f}_{\boldsymbol{m} - \boldsymbol{K} + \frac{\Delta\boldsymbol{k} - \Delta\boldsymbol{m}}{4}} \right\}(\boldsymbol{X}) \\ & \quad \times \mathrm{e}^{-\mathrm{i}\Delta\boldsymbol{m}\cdot\boldsymbol{Y}} \left\{ f_{\boldsymbol{l} - \frac{\Delta\boldsymbol{m}}{4}} \left(1 + \frac{\mathrm{i}}{4} \overleftrightarrow{\partial}_{\boldsymbol{x}} \cdot \overleftrightarrow{\partial}_{\boldsymbol{k}} \right) f_{\boldsymbol{m}-\boldsymbol{l} - \frac{\Delta\boldsymbol{m}}{4}} \right\}(\boldsymbol{Y}) \end{aligned} \tag{34}$$

where $\bar{f} = f + 1/V$ and $\boldsymbol{X}$, $\Delta\boldsymbol{x}$ and $\boldsymbol{Y}$, $\Delta\boldsymbol{y}$ are the centre and relative coordinates of $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ and $\boldsymbol{y}_1$, $\boldsymbol{y}_2$, respectively. The abbreviation $\partial_{\boldsymbol{x}} \partial_{\boldsymbol{k}} f_{\boldsymbol{K}}(\boldsymbol{X}) = \partial_{\boldsymbol{x}} \partial_{\boldsymbol{k}} f_{\boldsymbol{k}}(\boldsymbol{x}) \,|_{\boldsymbol{x}=\boldsymbol{X}, \boldsymbol{k}=\boldsymbol{K}}$ is also used.

Expanding again around $\bar{\boldsymbol{X}} = (\boldsymbol{X} + \boldsymbol{Y})/2$, the rhs of (34) up to the first order in the gradient becomes

$$V^3 \int \mathrm{d}\bar{\boldsymbol{X}}\, \mathrm{e}^{-\mathrm{i}\Delta\boldsymbol{k}\cdot\bar{\boldsymbol{X}}} \left\{ \bar{f}_{\boldsymbol{K}} \bar{f}_{m-\boldsymbol{K}} f_l f_{m-l} + \mathrm{i}y\left(\bar{f}_{\boldsymbol{K}}, \bar{f}_{m-\boldsymbol{K}}, f_l, f_{m-l}\right) \right\} (\bar{\boldsymbol{X}}) \tag{35}$$

where the gradient correction y is given as

$$\begin{aligned} y\left(\bar{f}_{\boldsymbol{k}}, \bar{f}_{\boldsymbol{k}'}, f_{\boldsymbol{q}}, f_{\boldsymbol{q}'}\right) \equiv \tfrac{1}{4} \Big\{ & \left(\bar{f}_{\boldsymbol{k}} \overleftrightarrow{\partial}_{\boldsymbol{x}} \cdot \overleftrightarrow{\partial}_{\boldsymbol{k}} \bar{f}_{\boldsymbol{k}'}\right) f_{\boldsymbol{q}} f_{\boldsymbol{q}'} - \bar{f}_{\boldsymbol{k}} \bar{f}_{\boldsymbol{k}'} \left(f_{\boldsymbol{q}} \overleftrightarrow{\partial}_{\boldsymbol{x}} \cdot \overleftrightarrow{\partial}_{\boldsymbol{k}} f_{\boldsymbol{q}'}\right) \\ & + (\partial_{\boldsymbol{x}} \bar{f}_{\boldsymbol{k}} \bar{f}_{\boldsymbol{k}'}) \cdot (\partial_{\boldsymbol{k}} f_{\boldsymbol{q}} f_{\boldsymbol{q}'}) - (\partial_{\boldsymbol{k}} \bar{f}_{\boldsymbol{k}} \bar{f}_{\boldsymbol{k}'}) \cdot (\partial_{\boldsymbol{x}} f_{\boldsymbol{q}} f_{\boldsymbol{q}'}) - 2(\partial_{\boldsymbol{x}} \cdot \partial_{\boldsymbol{k}} \bar{f}_{\boldsymbol{k}} \bar{f}_{\boldsymbol{k}'}) f_{\boldsymbol{q}} f_{\boldsymbol{q}'} \Big\}. \end{aligned} \tag{36}$$

The other terms in the last term of (33) can be processed in the same way, and the gradient expansion of the last term of (33) yields

$$\begin{aligned} \frac{\lambda^2 V^3}{\hbar} \sum_{l,m} \int_{t_1}^{t} \mathrm{d}s \int \mathrm{d}\boldsymbol{X}\, \mathrm{e}^{-\mathrm{i}\Delta\boldsymbol{k}\cdot\left\{\boldsymbol{X} + \boldsymbol{v}^{(2)}_{\boldsymbol{K},m-\boldsymbol{K}}(t-s)\right\}} \Big(& \cos\left\{\omega^{(2)}_{\boldsymbol{K},\boldsymbol{K},l,m}(t-s)\right\} F^{(2)}_{\boldsymbol{K},l,m}(\boldsymbol{X}, s) \\ & + \sin\left\{\omega^{(2)}_{\boldsymbol{K},\boldsymbol{K},l,m}(t-s)\right\} Y^{(2)}_{\boldsymbol{K},l,m}(\boldsymbol{X}, s)\Big). \end{aligned} \tag{37}$$

Here $\boldsymbol{v}^{(2)}_{\boldsymbol{K},m-\boldsymbol{K}} = (\boldsymbol{v}^{\mathrm{g}}_{\boldsymbol{K}} + \boldsymbol{v}^{\mathrm{g}}_{m-\boldsymbol{K}})/2$ appears from the expansion of $\omega^{(2)}_{\boldsymbol{k},\boldsymbol{q},l,m}$, and

$$F^{(2)}_{\boldsymbol{K},l,m} \equiv \bar{f}_{\boldsymbol{K}} \bar{f}_{m-\boldsymbol{K}} f_l f_{m-l} - f_{\boldsymbol{K}} f_{m-\boldsymbol{K}} \bar{f}_l \bar{f}_{m-l} \tag{38}$$

$$Y^{(2)}_{\boldsymbol{K},l,m} \equiv y\left(\bar{f}_{\boldsymbol{K}}, \bar{f}_{m-\boldsymbol{K}}, f_l, f_{m-l}\right) - y\left(f_{\boldsymbol{K}}, f_{m-\boldsymbol{K}}, \bar{f}_l, \bar{f}_{m-l}\right). \tag{39}$$

Combining (37) with (30), we obtain the QKE in the approximation of second order in the perturbation and first order in the gradient expansion as

$$\begin{aligned} & \left(\partial_t + \boldsymbol{v}^{\mathrm{g}}_{\boldsymbol{K}} \cdot \partial_{\boldsymbol{X}} - \frac{\lambda V}{\hbar} (\partial_{\boldsymbol{X}} n) \cdot \partial_{\boldsymbol{K}}\right) f_{\boldsymbol{K}}(\boldsymbol{X}, t) \\ & \quad = \frac{\lambda^2 V^3}{\hbar} \sum_{l,m} \int_{t_1}^{t} \mathrm{d}s \Big\{ \cos\left\{\omega^{(2)}_{\boldsymbol{K},\boldsymbol{K},l,m}(t-s)\right\} F^{(2)}_{\boldsymbol{K},l,m}(\boldsymbol{X} - \boldsymbol{v}^{(2)}_{\boldsymbol{K},m-\boldsymbol{K}}(t-s), s) \\ & \qquad + \sin\left\{\omega^{(2)}_{\boldsymbol{K},\boldsymbol{K},l,m}(t-s)\right\} Y^{(2)}_{\boldsymbol{K},l,m}(\boldsymbol{X} - \boldsymbol{v}^{(2)}_{\boldsymbol{K},m-\boldsymbol{K}}(t-s), s)\Big\}. \end{aligned} \tag{40}$$

Even after the gradient expansion, we can see the non-local contribution of the collision: the collision itself occurs locally but its contribution $F^{(2)}_{\boldsymbol{K},l,m}$ drifts with velocity $\boldsymbol{v}^{(2)}_{\boldsymbol{K},m-\boldsymbol{K}}$ into the remote WDF. In order to reduce (40) to the Markovian QKE, the adiabatic expansion is carried out. Assuming the time variation of $F^{(2)}$ and $Y^{(2)}$ to be small compared with the decay due to the summation over $\boldsymbol{l}, \boldsymbol{m}$ in (40), we expand them around the time t and integrate over s. Thus the adiabatic expansion of the rhs of (40) yields

$$\begin{aligned} \frac{\lambda^2 V^3}{\hbar} \sum_{l,m} \Bigg\{ & \left(\pi\delta(\omega^{(2)}_{\boldsymbol{K},\boldsymbol{K},l,m}) + \frac{\wp}{(\omega^{(2)}_{\boldsymbol{K},\boldsymbol{K},l,m})^2} \left(\partial_t + \boldsymbol{v}^{(2)}_{\boldsymbol{K},m-\boldsymbol{K}} \cdot \partial_{\boldsymbol{X}}\right)\right) F^{(2)}_{\boldsymbol{K},l,m}(\boldsymbol{X}, t) \\ & - \left(\frac{\wp}{\omega^{(2)}_{\boldsymbol{K},\boldsymbol{K},l,m}} - \delta'(\omega^{(2)}_{\boldsymbol{K},\boldsymbol{K},l,m}) \left(\partial_t + \boldsymbol{v}^{(2)}_{\boldsymbol{K},m-\boldsymbol{K}} \cdot \partial_{\boldsymbol{X}}\right)\right) Y^{(2)}_{\boldsymbol{K},l,m}(\boldsymbol{X}, t) \Bigg\}. \end{aligned} \tag{41}$$

The drift of $F^{(2)}$ and $Y^{(2)}$ will be seen from the operation $(\partial_t + \boldsymbol{v}^{(2)} \cdot \partial_{\boldsymbol{X}})$. The drift of $Y^{(2)}$, however, becomes higher order in gradient and will be neglected. If we take only the energy conserving term $\delta(\omega^{(2)})F^{(2)}$, equation (40) is reduced to the usual Boltzmann equation. A similar expression of the collision integral is obtained in [13], but the drift of the collision term is not considered there because only the leading order of the Markovian approximation is taken.

4. Conservation laws

In the field of kinetic theory, it is important to show that the conservation laws are kept by the obtained kinetic equation. Usually the local conservation laws are derived by making the time derivative of the conserved quantities into the form of the gradient [13, 14], but, since we have the Lagrangian, the conserved currents can directly be derived from the symmetries of the Lagrangian with the use of Noether's theorem. The currents thus derived are expressed as functionals of z and the conservation laws are explicitly confirmed by QKE (33).

4.1. Local conservation of number density

For definiteness, we choose the dispersion relation $\epsilon_k = \hbar^2 \boldsymbol{k}^2/2m$ in this section. From (6), the Lagrangian density is given in this case as

$$\mathcal{L}(\boldsymbol{x}) = \frac{\mathrm{i}\hbar}{2}\hat{\psi}^\dagger(\boldsymbol{x}) \overset{\leftrightarrow}{\partial}_t \hat{\psi}(\boldsymbol{x}) - \frac{\hbar^2}{2m}\partial_{\boldsymbol{x}}\hat{\psi}^\dagger(\boldsymbol{x}) \cdot \partial_{\boldsymbol{x}}\hat{\psi}(\boldsymbol{x}) - \frac{\lambda V}{4}\hat{\psi}^\dagger(\boldsymbol{x})\hat{\psi}^\dagger(\boldsymbol{x})\hat{\psi}(\boldsymbol{x})\hat{\psi}(\boldsymbol{x}). \tag{42}$$

Then from the invariance under the global phase shift of the field, the Noether theorem gives us the conserved current

$$j_0(\boldsymbol{x}) = \left\langle \hat{\psi}^\dagger(\boldsymbol{x})\hat{\psi}(\boldsymbol{x}) \right\rangle \tag{43}$$

$$= \sum_{k,q} z_{k,q} \frac{\mathrm{e}^{\mathrm{i}(k-q)\cdot x}}{V} = \sum_{K} f_K(\boldsymbol{x}) = n(\boldsymbol{x}) \tag{44}$$

$$\boldsymbol{j}(\boldsymbol{x}) = -\frac{\mathrm{i}\hbar}{2m}\left\{ \left\langle \hat{\psi}^\dagger(\boldsymbol{x})\partial_{\boldsymbol{x}}\hat{\psi}(\boldsymbol{x}) \right\rangle - \left\langle (\partial_{\boldsymbol{x}}\hat{\psi}^\dagger(\boldsymbol{x}))\hat{\psi}(\boldsymbol{x}) \right\rangle \right\} \tag{45}$$

$$= \frac{\hbar}{2m}\sum_{k,q}(\boldsymbol{k}+\boldsymbol{q}) z_{k,q} \frac{\mathrm{e}^{\mathrm{i}(k-q)\cdot x}}{V} = \sum_{K} \boldsymbol{v}^{\mathrm{g}}_{K} f_K(\boldsymbol{x}) \tag{46}$$

which satisfies $\partial_\mu j_\mu = \partial_t j_0 + \partial_{\boldsymbol{x}} \cdot \boldsymbol{j} = 0$, i.e. the local conservation of the particle density. As shown in (44) and (46), this current can simply be expressed as a function of the WDF since it involves no interaction contribution.

The conservation relation can explicitly be checked using the kinetic equation (33). From (33), the time derivative of the number density (44) is rewritten as

$$\partial_t n(\boldsymbol{x}) = \sum_{k,q} \frac{\epsilon_k - \epsilon_q}{\mathrm{i}\hbar} z_{k,q} \frac{\mathrm{e}^{\mathrm{i}(k-q)\cdot x}}{V} \tag{47}$$

$$= -\sum_{k,q} \mathrm{i}(\boldsymbol{k}-\boldsymbol{q}) \cdot \boldsymbol{v}^{\mathrm{g}}_{\frac{k+q}{2}} z_{k,q} \frac{\mathrm{e}^{\mathrm{i}(k-q)\cdot x}}{V} = -\partial_{\boldsymbol{x}} \cdot \boldsymbol{j}(\boldsymbol{x}) \tag{48}$$

and thus the local conservation of the number density is confirmed. In the rhs of (47), the terms from the first- and second-order corrections in (33) vanish by symmetry after the summation over $\boldsymbol{k}$ and $\boldsymbol{q}$.

4.2. Conservation of energy and momentum densities

Applying the Noether theorem to the invariance under the translation in space and time, the energy–momentum tensor satisfying $\partial_\nu T_{\mu\nu} = 0$ is obtained as

$$T_{00} = \frac{\mathrm{i}\hbar}{2}\left\langle \hat{\psi}^\dagger(\boldsymbol{x}) \overset{\leftrightarrow}{\partial}_t \hat{\psi}(\boldsymbol{x}) \right\rangle - \langle\, \mathcal{L}(\boldsymbol{x}) \,\rangle = \varepsilon \tag{49}$$

$$T_{0i} = -\frac{\hbar^2}{2m}\left(\left\langle (\partial_t\hat{\psi}^\dagger(\boldsymbol{x}))\partial_i\hat{\psi}(\boldsymbol{x})\right\rangle + \left\langle (\partial_i\hat{\psi}^\dagger(\boldsymbol{x}))\partial_t\hat{\psi}(\boldsymbol{x})\right\rangle\right) = j_{\varepsilon,i} \tag{50}$$

$$T_{i0} = \frac{\mathrm{i}\hbar}{2}\left\langle \hat{\psi}^\dagger(\boldsymbol{x}) \overset{\leftrightarrow}{\partial}_i \hat{\psi}(\boldsymbol{x})\right\rangle = -P_i \tag{51}$$

$$T_{ij} = -\frac{\hbar^2}{2m}\left(\left\langle (\partial_i\hat{\psi}^\dagger(\boldsymbol{x}))\partial_j\hat{\psi}(\boldsymbol{x})\right\rangle + \left\langle (\partial_j\hat{\psi}^\dagger(\boldsymbol{x}))\partial_i\hat{\psi}(\boldsymbol{x})\right\rangle\right) - \delta_{ij}\langle\,\mathcal{L}(\boldsymbol{x})\,\rangle = -\mathcal{Q}_{ij} \tag{52}$$

where ε and $\boldsymbol{P}$ are the energy and momentum densities, respectively, while $\boldsymbol{j}_\varepsilon$ and $\mathcal{Q}$ are corresponding currents.

With the use of Fourier transformation, the momentum density can simply be expressed as a function of z:

$$\boldsymbol{P}(\boldsymbol{x}) = \sum_{\boldsymbol{k},\boldsymbol{q}} \hbar\frac{\boldsymbol{k}+\boldsymbol{q}}{2} z_{\boldsymbol{k},\boldsymbol{q}} \mathrm{e}^{\mathrm{i}(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{x}} = \sum_{\boldsymbol{K}} \hbar\boldsymbol{K} f_{\boldsymbol{K}}(\boldsymbol{x}) \tag{53}$$

but the energy density and the currents receive the interaction contributions and are written as

$$\varepsilon(\boldsymbol{x}) = \sum_{\boldsymbol{k},\boldsymbol{q}} \frac{\mathrm{e}^{\mathrm{i}(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{x}}}{V}\left(\frac{\hbar^2}{2m}(\boldsymbol{k}\cdot\boldsymbol{q}) z_{\boldsymbol{k},\boldsymbol{q}} + \sum_{l,m}\frac{\lambda}{4}\left\langle \hat{\psi}^\dagger_{\boldsymbol{q}-\boldsymbol{m}}\hat{\psi}^\dagger_{\boldsymbol{m}}\hat{\psi}_{\boldsymbol{k}-\boldsymbol{l}}\hat{\psi}_{\boldsymbol{l}}\right\rangle\right) \tag{54}$$

$$\boldsymbol{j}_\varepsilon(\boldsymbol{x}) = \frac{\hbar^2}{2\mathrm{i}m}\sum_{\boldsymbol{k},\boldsymbol{q}} \frac{\mathrm{e}^{\mathrm{i}(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{x}}}{V}\left\{\boldsymbol{k}\left\langle (\partial_t\hat{\psi}^\dagger_{\boldsymbol{q}})\hat{\psi}_{\boldsymbol{k}}\right\rangle - \boldsymbol{q}\left\langle \hat{\psi}^\dagger_{\boldsymbol{q}}\partial_t\hat{\psi}_{\boldsymbol{k}}\right\rangle\right\} \tag{55}$$

$$\mathcal{Q}(\boldsymbol{x}) = \sum_{\boldsymbol{k},\boldsymbol{q}} \frac{\mathrm{e}^{\mathrm{i}(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{x}}}{V}\left\{\frac{\hbar^2}{2m}(\boldsymbol{q}\otimes\boldsymbol{k}+\boldsymbol{k}\otimes\boldsymbol{q})\, z_{\boldsymbol{k},\boldsymbol{q}} + \frac{\mathrm{i}\hbar}{2}\mathcal{I}\left\langle \hat{\psi}^\dagger_{\boldsymbol{q}} \overset{\leftrightarrow}{\partial}_t \hat{\psi}_{\boldsymbol{k}}\right\rangle\right\} - \mathcal{I}\varepsilon(\boldsymbol{x}) \tag{56}$$

where $(\boldsymbol{a}\otimes\boldsymbol{b})_{ij} = a_i b_j$ and $\mathcal{I}$ is a unit tensor. Using the relation

$$\left\langle \hat{\psi}^\dagger_{\boldsymbol{q}}\partial_t\hat{\psi}_{\boldsymbol{k}}\right\rangle = \left\langle (\partial_t\hat{\psi}^\dagger_{\boldsymbol{k}})\hat{\psi}_{\boldsymbol{q}}\right\rangle^* = \frac{\epsilon_{\boldsymbol{k}}}{\mathrm{i}\hbar} z_{\boldsymbol{k},\boldsymbol{q}} + \frac{\lambda}{2\mathrm{i}\hbar}\sum_{l,m}\left\langle \hat{\psi}^\dagger_{\boldsymbol{q}}\hat{\psi}^\dagger_{\boldsymbol{l}}\hat{\psi}_{\boldsymbol{k}+\boldsymbol{m}}\hat{\psi}_{\boldsymbol{l}-\boldsymbol{m}}\right\rangle \tag{57}$$

which is obtained from the exact EoM of the field operator, the currents (55) and (56) become

$$\boldsymbol{j}_\varepsilon(\boldsymbol{x}) = \frac{\hbar^2}{2m}\sum_{\boldsymbol{k},\boldsymbol{q}} \frac{\mathrm{e}^{\mathrm{i}(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{x}}}{V}\left\{\frac{\hbar}{2m}\left(\boldsymbol{q}^2\boldsymbol{k}+\boldsymbol{k}^2\boldsymbol{q}\right) z_{\boldsymbol{k},\boldsymbol{q}} + \sum_{l,m}\frac{\lambda}{2\hbar}(\boldsymbol{l}+\boldsymbol{m})\left\langle \hat{\psi}^\dagger_{\boldsymbol{q}-\boldsymbol{m}}\hat{\psi}^\dagger_{\boldsymbol{m}}\hat{\psi}_{\boldsymbol{k}-\boldsymbol{l}}\hat{\psi}_{\boldsymbol{l}}\right\rangle\right\} \tag{58}$$

$$\mathcal{Q}(\boldsymbol{x}) = \sum_{\boldsymbol{k},\boldsymbol{q}} \frac{\mathrm{e}^{\mathrm{i}(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{x}}}{V}\left(\frac{\hbar^2}{2m}\left\{\boldsymbol{q}\otimes\boldsymbol{k}+\boldsymbol{k}\otimes\boldsymbol{q}+\mathcal{I}\frac{(\boldsymbol{k}-\boldsymbol{q})^2}{2}\right\} z_{\boldsymbol{k},\boldsymbol{q}} + \sum_{l,m}\frac{\lambda}{4}\mathcal{I}\left\langle \hat{\psi}^\dagger_{\boldsymbol{q}-\boldsymbol{m}}\hat{\psi}^\dagger_{\boldsymbol{m}}\hat{\psi}_{\boldsymbol{k}-\boldsymbol{l}}\hat{\psi}_{\boldsymbol{l}}\right\rangle\right). \tag{59}$$

In order to express the energy density (54) and the currents (58) and (59) as functionals of the WDF, the four-point function must be written in terms of z. For this, the four-point function is first calculated as a functional of J by a perturbation expansion with the propagator derived in section 3.2, and then the functional is evaluated at $J = J[z]$ using the inverted relation, equation (27) with the last term in (33) subtracted and multiplied by $\hbar$. Thus after some calculations, equations (54), (58) and (59) up to O(λ) yield

$$\varepsilon(\boldsymbol{x}) = \sum_{\boldsymbol{K}}\frac{\hbar^2}{2m}\left(\boldsymbol{K}^2 + \frac{\triangle}{4}\right) f_{\boldsymbol{K}}(\boldsymbol{x}) + \frac{\lambda V}{2} n^2(\boldsymbol{x}) \tag{60}$$

$$\boldsymbol{j}_\varepsilon(\boldsymbol{x}) = \sum_{\boldsymbol{K}}\frac{\hbar^2}{2m}\left\{\left(\boldsymbol{K}^2 - \frac{\triangle}{4}\right)\boldsymbol{v}^{\mathrm{g}}_{\boldsymbol{K}} + \frac{1}{2}(\boldsymbol{v}^{\mathrm{g}}_{\boldsymbol{K}}\cdot\partial_{\boldsymbol{x}})\partial_{\boldsymbol{x}}\right\} f_{\boldsymbol{K}}(\boldsymbol{x}) + \lambda V n(\boldsymbol{x})\boldsymbol{j}(\boldsymbol{x}) \tag{61}$$

$$\mathcal{Q}(\boldsymbol{x}) = \sum_{\boldsymbol{K}} \left\{ \hbar \boldsymbol{K} \otimes \boldsymbol{v}_{\boldsymbol{K}}^{\mathrm{g}} + \frac{\hbar^2}{4m} \left(\partial_{\boldsymbol{x}} \otimes \partial_{\boldsymbol{x}} - \mathcal{I} \triangle \right) \right\} f_{\boldsymbol{K}}(\boldsymbol{x}) + \frac{\lambda V}{2} n^2(\boldsymbol{x}) \mathcal{I} \tag{62}$$

where $\boldsymbol{j}$ is the number current defined in (45).

The second-order corrections to these quantities become somewhat complicated and cannot be expressed by local functions of the WDF. Similarly as in section 3.4, the tadpole-diagram contributions to the four-point function cancel with contributions from the perturbative corrections of $J[z]$. Only the non-tadpole diagrams are retained, and they are evaluated at lowest-order $J[z]$, i.e. equation (23) in which $z^{(0)}$s are replaced by zs. Hence the second-order corrections can be written as

$$\varepsilon^{(2)}(\boldsymbol{x}) = \frac{\mathrm{i}\lambda^2}{4\hbar} \sum_{\boldsymbol{k},\boldsymbol{q},\boldsymbol{l},\boldsymbol{m}} \frac{\mathrm{e}^{\mathrm{i}(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{x}}}{V} \int_{t_{\mathrm{I}}}^{t} \mathrm{d}s \, \mathrm{e}^{\mathrm{i}\omega^{(2)}_{\boldsymbol{q},\boldsymbol{k},\boldsymbol{l},\boldsymbol{m}}(t-s)} Z^{(2)}_{\boldsymbol{q},\boldsymbol{k},\boldsymbol{l},\boldsymbol{m}}(s) \tag{63}$$

$$\boldsymbol{j}^{(2)}_{\varepsilon}(\boldsymbol{x}) = \frac{\mathrm{i}}{4} \left(\frac{\lambda}{\hbar} \right)^2 \sum_{\boldsymbol{k},\boldsymbol{q},\boldsymbol{l},\boldsymbol{m}} \frac{\mathrm{e}^{\mathrm{i}(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{x}}}{V} \frac{\hbar}{m} (\boldsymbol{q}+\boldsymbol{l}) \int_{t_{\mathrm{I}}}^{t} \mathrm{d}s \, \mathrm{e}^{\mathrm{i}\omega^{(2)}_{\boldsymbol{q},\boldsymbol{k},\boldsymbol{l},\boldsymbol{m}}(t-s)} Z^{(2)}_{\boldsymbol{q},\boldsymbol{k},\boldsymbol{l},\boldsymbol{m}}(s) \tag{64}$$

$$\mathcal{Q}^{(2)}(\boldsymbol{x}) = \varepsilon^{(2)}(\boldsymbol{x}) \mathcal{I}. \tag{65}$$

These corrections (63)–(65) can be rewritten in terms of the WDF by (8), but such expressions seem to be of little use.

Though the calculations are tedious, it is not difficult to see that the above-derived currents satisfy the local conservation laws via the QKE (33). From (53), (60) and (63), the time derivatives of densities can be calculated by the use of (33), and the divergences of currents are from (61), (62), (64) and (65). After some relabelling of the summation indices, in each order of the perturbation, we can make the divergences of momentum and energy currents precisely the negative of the time derivatives of the corresponding densities.

The above order-by-order realization of the conservation is not a trivial result from the Noether theorem. As pointed out in [9, 10], the process of inversion is non-perturbative and the diagrams taken into account by the inversion depend on how the source is introduced. Hence the QKE has different forms for different sources [7]. Of course in the full-order calculation, the conservation laws must be recovered whatever the source is, but in the finite-order calculation, the order-by-order conservation is not generally realized.

5. Discussions

We have derived the QKE in the case of the inhomogeneous initial condition. The inversion method in [6] was utilized and the main result is the QKE (33), which is non-local in the WDF and non-Markovian. The non-local feature of the QKE is taken into account in [13, 14] by the operator formalism or in [8] by the counter-term method, but the non-Markovian property is not considered therein. As the result, the Markovian approximation of the QKE reveals a new feature: the drift in the collision term. The higher-order corrections and the effect of the initial correlations can be incorporated in the same way as in [7], and, if we go beyond the perturbation theory, the T-matrix approximation or the dynamical screening will be incorporated similarly as in the GKB formalism.

As shown in section 3, we have just calculated the (Fourier transform of the) WDF and no ansatz on the form of the Green function is required. Moreover, the advantage of using the inversion method can also be seen in the derivation of conservation laws in section 4. The evaluation of the conserved current can be carried out quite systematically: The current first calculated as a functional of source J is written in terms of the WDF z or f by evaluating the functional at $J = J[z]$, i.e. the inverted relation. Thus the vanishing of the tadpole contribution to the currents is automatic in our formalism, while it will not be in the GKB formalism.

As mentioned in section 3.4, we can explicitly show that the second-order diagrams with tadpoles vanish after the inversion. This fact suggests that some of the diagrams for $z[J]$ can be eliminated if our interest is in the inverted relation. Moreover, as pointed out above, the four-point functions in the conserved currents can be calculated as functionals of z from the diagrams without tadpoles in which the source J is evaluated at the lowest order of inversion $J[z]$. This suggests that the expectation value of a physical quantity is expressed as a functional of z by discarding some part of the diagrams. Such properties are investigated in the case of a normal probing source [10, 15, 16], and it is known that the modified two-particle irreducible diagrams are enough when a composite two-particle operator is probed [16]. The diagrammatic discussion using our extended probing source is under construction.

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